(b) Use this function to predict the cost of driving 2000 km per month.

(c) What does the slope of the function represent?

(d) What is the monthly cost if she does not drive her car at all? Is it reasonable?

(e) Why is a linear function a suitable model in this situation?

1.2 THE LIMIT OF A FUNCTION

We saw in the first section how limits arise in trying to find a tangent line to a curve. Later in this chapter we will see that limits also arise in computing velocities and other rates of change. In fact, limits are basic to all of calculus and so in this section we look at limits in general and methods for calculating them.

We begin by investigating the behaviour of the function

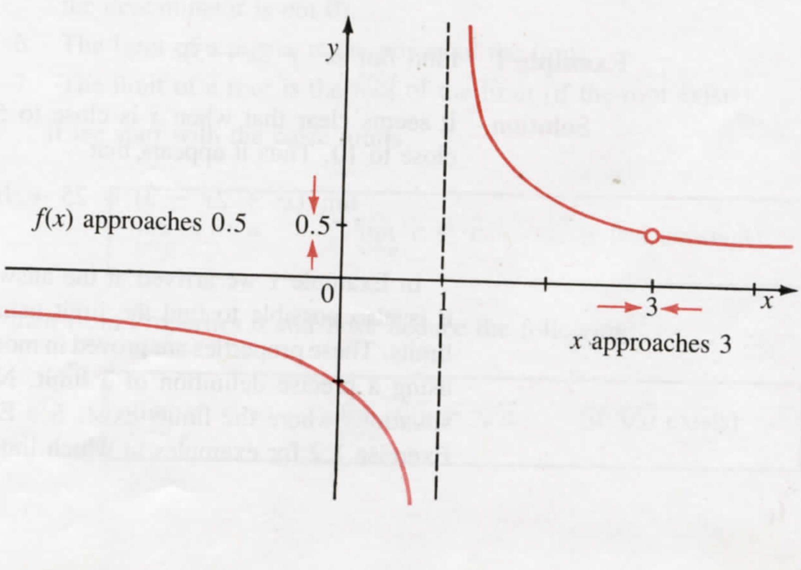
$$f(x) = \frac{x - 3}{x^2 - 4x + 3}$$

when x is near 3. The following table gives values of $f(x)$ for values of x approaching 3 (but not equal to 3).

$x < 3$	$f(x)$
2.5	0.666 667
2.9	0.526 316
2.99	0.502 513
2.999	0.500 250
2.9999	0.500 025

$x > 3$	$f(x)$
3.5	0.400 000
3.1	0.476 190
3.01	0.497 512
3.001	0.499 750
3.0001	0.499 975

The open circle at $\left(3, \frac{1}{2}\right)$ indicates that the function is not defined when $x = 3$.



From the table and the graph of f, we see that when x is close to 3 (on either side of 3), $f(x)$ is close to 0.5. In fact, it appears that we can make the values of $f(x)$ as close as we like to 0.5 by taking x close enough to 3. We express this by saying

$$\text{“the limit of } \frac{x-3}{x^2-4x+3} \text{ as } x \text{ approaches 3 is equal to } \tfrac{1}{2}\text{”}$$

and by writing

$$\lim_{x\to 3} \frac{x-3}{x^2-4x+3} = \frac{1}{2}$$

In general, we have the following definition of the limit of a function.

> We write $\lim_{x\to a} f(x) = L$
>
> and say
>
> “the limit of $f(x)$, as x approaches a, equals L”
>
> if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a, but not equal to a.

Roughly speaking, this says that the values of $f(x)$ become closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$.

Notice the phrase “but $x \neq a$” in the definition of a limit. This means that in finding the limit of $f(x)$ as x approaches a, we need never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. (The function f considered before the definition is not defined at $x = 3$.) The only thing that matters is how f is defined *near a*.

Example 1 Find $\lim_{x\to 5} (x^2 + 2x - 3)$.

Solution It seems clear that when x is close to 5, x^2 is close to 25 and $2x$ is close to 10. Thus it appears that

$$\lim_{x\to 5} (x^2 + 2x - 3) = 25 + 10 - 3 = 32$$

In Example 1 we arrived at the answer by intuitive reasoning, but it is also possible to find the limit using the following properties of limits. These properties are proved in more advanced courses in calculus using a precise definition of a limit. Notice that they apply only in situations where the limits exist. See Example 8 and Question 12 in Exercise 1.2 for examples in which limits do not exist.

Properties of Limits

Suppose that the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

both exist and let c be a constant. Then

1 $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ if n is a positive integer

7 $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ if the root on the right side exists

These seven properties of limits can be stated verbally as follows.

1 The limit of a sum is the sum of the limits.
2. The limit of a difference is the difference of the limits.
3. The limit of a constant times a function is the constant times the limit of the function.
4. The limit of a product is the product of the limits.
5. The limit of a quotient is the quotient of the limits (if the limit of the denominator is not 0).
6. The limit of a power is the power of the limit.
7 The limit of a root is the root of the limit (if the root exists).

If we start with the basic limits

$$\lim_{x \to a} x = a \qquad \lim_{x \to a} c = c \qquad (c \text{ is a constant})$$

then from Properties 6 and 7 we deduce the following:

$$\lim_{x \to a} x^n = a^n \qquad \lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \qquad (\text{if } \sqrt[n]{a} \text{ exists})$$

Using these limits, together with the seven properties of limits, we can compute limits of more complicated functions. First we return to the limit of Example 1

Example 2 Find $\lim_{x\to 5} (x^2 + 2x - 3)$ using the properties of limits.

Solution

$$\begin{aligned}
\lim_{x\to 5} (x^2 + 2x - 3) &= \lim_{x\to 5} x^2 + \lim_{x\to 5} 2x - \lim_{x\to 5} 3 && \text{(Properties 1 and 2)}\\
&= \lim_{x\to 5} x^2 + 2\lim_{x\to 5} x - \lim_{x\to 5} 3 && \text{(Property 3)}\\
&= 5^2 + 2(5) - 3\\
&= 32
\end{aligned}$$

Example 3 Evaluate using the properties of limits.

(a) $\displaystyle\lim_{x\to 1} \frac{x^4 - 5x^2 + 1}{x + 2}$ (b) $\displaystyle\lim_{x\to 3} \sqrt{x^2 + x}$

Solution (a)

$$\begin{aligned}
\lim_{x\to 1} \frac{x^4 - 5x^2 + 1}{x + 2} &= \frac{\lim_{x\to 1} (x^4 - 5x^2 + 1)}{\lim_{x\to 1} (x + 2)} && \text{(Property 5)}\\
&= \frac{\lim_{x\to 1} x^4 - 5\lim_{x\to 1} x^2 + \lim_{x\to 1} 1}{\lim_{x\to 1} x + \lim_{x\to 1} 2} && \text{(Properties 2, 3, and 1)}\\
&= \frac{1^4 - 5(1)^2 + 1}{1 + 2}\\
&= -1
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to 3} \sqrt{x^2 + x} &= \sqrt{\lim_{x\to 3} (x^2 + x)} && \text{(Property 7)}\\
&= \sqrt{\lim_{x\to 3} x^2 + \lim_{x\to 3} x} && \text{(Property 1)}\\
&= \sqrt{3^2 + 3}\\
&= \sqrt{12}\\
&= 2\sqrt{3}
\end{aligned}$$

Notice that if we let

$$f(x) = \frac{x^4 - 5x^2 + 1}{x + 2}$$

then

$$f(1) = \frac{1^4 - 5(1)^2 + 1}{1 + 2} = -1$$

and so we would have got the right answer in Example 3(a) by substituting 1 for x:

$$\lim_{x \to 1} f(x) = f(1)$$

Similarly, direct substitution provides the correct answer in Example 3(b):

$$\text{If } g(x) = \sqrt{x^2 + x}, \text{ then } \lim_{x \to 3} g(x) = g(3).$$

Functions with this property, that is,

$$\lim_{x \to a} f(x) = f(a)$$

are called **continuous at *a***. The geometric properties of such functions will be studied in the next section.

Using the properties of limits, it can be shown that many familiar functions are continuous. Recall that a **polynomial** is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \quad + a_1 x + a_0$$

where $a_0, a_1, \quad , a_n$ are constants. A **rational function** is a ratio of polynomials.

(a) Any polynomial P is continuous at every number; that is,

$$\lim_{x \to a} P(x) = P(a)$$

(b) Any rational function $f(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials, is continuous at every number a such that $Q(a) \neq 0$; that is,

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)} \qquad Q(a) \neq 0$$

For instance, we could rework the solution to Example 2 as follows:

$$f(x) = x^2 + 2x - 3 \text{ is a polynomial, so it is continuous at } 5$$

and therefore,

$$\lim_{x \to 5} (x^2 + 2x - 3) = f(5) = 5^2 + 2(5) - 3 = 32$$

Not all limits can be evaluated by direct substitution, however, as the following examples illustrate.

Example 4 Evaluate $\lim_{x\to 4} \dfrac{x^2 - 16}{x - 4}$

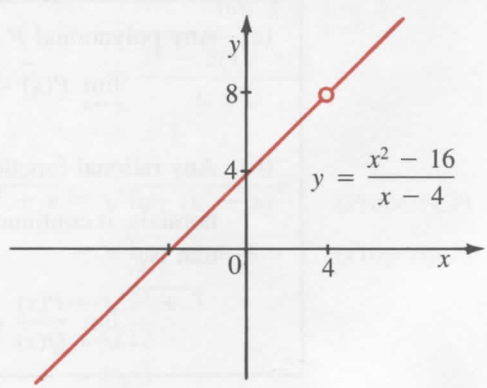
Solution Let

$$f(x) = \frac{x^2 - 16}{x - 4}$$

We cannot find the limit by substituting $x = 4$ because $f(4)$ is not defined ($\frac{0}{0}$ is meaningless). Remember that the definition of $\lim_{x\to a} f(x)$ says that we consider values of x that are close to a but not equal to a. Therefore in this example we have $x \neq 4$, so we can factor the numerator as a difference of squares and write

$$\begin{aligned} \lim_{x\to 4} \frac{x^2 - 16}{x - 4} &= \lim_{x\to 4} \frac{(x-4)(x+4)}{x-4} \\ &= \lim_{x\to 4} (x+4) \\ &= 4 + 4 \\ &= 8 \end{aligned}$$

Notice that in Example 4 we replaced the given rational function by a continuous function $[g(x) = x + 4]$ that is equal to $f(x)$ for $x \neq 4$. This is illustrated by the graph of f



Example 5 Find $\lim_{x\to 2} \dfrac{x^3 - 8}{x^2 - 3x + 2}$

Solution Notice that we cannot substitute $x = 2$ since we would obtain $\frac{0}{0}$. We replace the given rational function by a rational function that is continuous at 2. To do this, we factor the numerator by using the formula for a difference of cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

with $a = x$ and $b = 2$. Then

$$\begin{aligned}\lim_{x\to 2}\frac{x^3-8}{x^2-3x+2} &= \lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x-2)(x-1)}\\ &= \lim_{x\to 2}\frac{x^2+2x+4}{x-1}\\ &= \frac{2^2+2(2)+4}{2-1}\\ &= 12\end{aligned}$$

Example 6 Find $\lim\limits_{h\to 0}\dfrac{(2+h)^2-4}{h}$

Solution Again we cannot compute the limit by letting $h = 0$, so we first simplify the numerator:

$$\begin{aligned}\lim_{h\to 0}\frac{(2+h)^2-4}{h} &= \lim_{h\to 0}\frac{(4+4h+h^2)-4}{h}\\ &= \lim_{h\to 0}\frac{4h+h^2}{h}\\ &= \lim_{h\to 0}(4+h)\\ &= 4\end{aligned}$$

Example 7 Evaluate $$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}.$$

Solution Here the algebraic simplification consists of rationalizing the numerator, that is, multiplying numerator and denominator by the conjugate radical $\sqrt{x+1}+1$

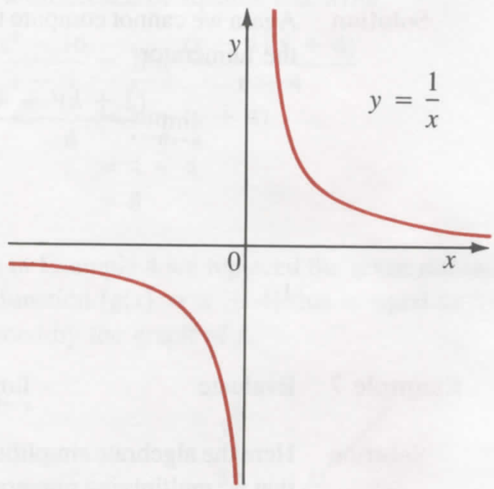
Do not expand the denominator

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0}\left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)\\ &= \lim_{x\to 0}\frac{(x+1)-1}{x(\sqrt{x+1}+1)}\\ &= \lim_{x\to 0}\frac{x}{x(\sqrt{x+1}+1)}\\ &= \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}\\ &= \frac{\lim\limits_{x\to 0}1}{\sqrt{\lim\limits_{x\to 0}(x+1)}+\lim\limits_{x\to 0}1}\\ &= \frac{1}{\sqrt{0+1}+1}\\ &= \frac{1}{2}\end{aligned}$$

Example 8 Show that $\lim_{x\to 0} \frac{1}{x}$ does not exist.

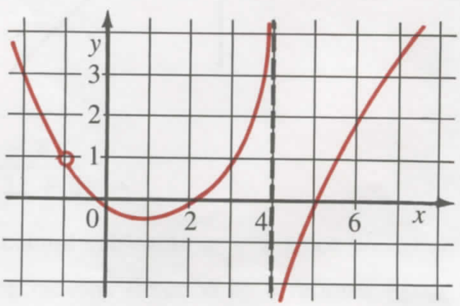
Solution As x approaches 0 through positive values, $\frac{1}{x}$ becomes very large. As x approaches 0 through negative values, $\frac{1}{x}$ becomes very large negative. We see from the graph of $y = \frac{1}{x}$ that the values of y do not approach any number as x approaches 0. Therefore

$$\lim_{x\to 0} \frac{1}{x} \text{ does not exist}$$



EXERCISE 1.2

A 1. Use the given graph of f to state the value of the limit, if it exists.



(a) $\lim_{x\to 3} f(x)$ (b) $\lim_{x\to 2} f(x)$ (c) $\lim_{x\to -1} f(x)$ (d) $\lim_{x\to 4} f(x)$

2. State the value of each limit.

(a) $\lim_{x \to 2} x^3$ (b) $\lim_{x \to \pi} x$

(c) $\lim_{x \to 8} 3$ (d) $\lim_{x \to 4} \sqrt{x}$

(e) $\lim_{x \to k} x^6$ (f) $\lim_{x \to 0} \pi$

B **3.** Use the properties of limits to evaluate the following.

(a) $\lim_{x \to 1} (3x - 7)$ (b) $\lim_{x \to -1} (2x^2 - 5x + 3)$

(c) $\lim_{x \to 2} (x^3 + x^2 - 2x - 8)$ (d) $\lim_{x \to -2} (x^2 + 5x + 3)^6$

(e) $\lim_{x \to 0} \dfrac{x - 1}{x + 1}$ (f) $\lim_{x \to 4} \dfrac{x^2 + 2x - 3}{x^2 + 2}$

(g) $\lim_{t \to 2} \dfrac{t^4 - 3t + 1}{t^2(t - 1)^3}$ (h) $\lim_{u \to -4} \sqrt{u^4 + 2u^2}$

(i) $\lim_{x \to 5} \sqrt[3]{x^2 + 2x - 8}$ (j) $\lim_{t \to 3} \left(2t^2 + \sqrt{\dfrac{6 + t}{4 - t}}\right)$

4. Find the following limits.

(a) $\lim_{x \to -2} \dfrac{x + 2}{x^2 - 4}$ (b) $\lim_{x \to 1} \dfrac{x^2 - 3x + 2}{x - 1}$

(c) $\lim_{x \to 3} \dfrac{x^2 - 2x - 3}{x^2 - 4x + 3}$ (d) $\lim_{x \to -2} \dfrac{2x^2 + 5x + 2}{x^2 - 2x - 8}$

(e) $\lim_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1}$ (f) $\lim_{x \to -3} \dfrac{x + 3}{x^3 + 27}$

(g) $\lim_{x \to 9} \dfrac{x - 9}{\sqrt{x} - 3}$ (h) $\lim_{x \to 2} \dfrac{\frac{1}{x} - \frac{1}{2}}{x - 2}$

5. Evaluate the following.

(a) $\lim_{h \to 0} \dfrac{(4 + h)^3 - 64}{h}$ (b) $\lim_{h \to 0} \dfrac{(h - 2)^2 - 4}{h}$

(c) $\lim_{h \to 0} \dfrac{\frac{1}{1 + h} - 1}{h}$ (d) $\lim_{h \to 0} \dfrac{(2 + h)^4 - 16}{h}$

(e) $\lim_{h \to 0} \dfrac{\sqrt{9 + h} - 3}{h}$ (f) $\lim_{h \to 0} \dfrac{\frac{1}{(2 + h)^2} - \frac{1}{4}}{h}$

6. Find the following limits, if they exist.

(a) $\lim_{x \to 3} \dfrac{1}{(x - 3)^2}$ (b) $\lim_{x \to -8} \dfrac{x^2 + 16x + 64}{x + 8}$

(c) $\lim_{x \to 1} \dfrac{x^4 - 1}{x - 1}$ (d) $\lim_{x \to -1} \dfrac{x - 1}{x^2 - 1}$

(e) $\lim_{x \to 1} \dfrac{x^2 + x - 2}{x^2 - 2x + 1}$ (f) $\lim_{x \to -2} \dfrac{x^2 - x - 2}{x^2 + 3x + 2}$

(g) $\lim_{x \to 3} \dfrac{x^{-2} - 3^{-2}}{x - 3}$ (h) $\lim_{x \to 4} \dfrac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{2}}{x - 4}$

(i) $\lim_{x \to 1} \dfrac{x^3 - 1}{x^3 - x^2 - 4x + 4}$ (j) $\lim_{x \to 1} \dfrac{x - 1}{\sqrt{x} - x}$

7. (a) Use your calculator to evaluate $f(x) = (1 + x)^{\frac{1}{x}}$ correct to six decimal places for $x = 1, 0.1, 0.01, 0.001, 0.000\ 1, 0.000\ 01, 0.000\ 001$, and $0.000\ 000\ 1$

(b) Estimate the value of the limit

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$$

to five decimal places.

8. (a) Use your calculator to evaluate $g(x) = \dfrac{2^x - 1}{x}$ correct to four decimal places for $x = 1, 0.1, 0.01, 0.001, 0.000\ 1$

(b) Estimate the value of the limit

$$\lim_{h \to 0} \frac{2^h - 1}{h}$$

to three decimal places.

C **9.** Evaluate the following limits.

(a) $\lim_{x \to 8} \dfrac{x - 8}{\sqrt[3]{x} - 2}$ (b) $\lim_{x \to 2} \dfrac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$

10. If $f(x) = 2x + 3$, show that

$$|f(x) - 7| < 0.01 \quad \text{if} \quad |x - 2| < 0.005$$

11. How close to 1 do we have to take x so that $\dfrac{16x^2 - 1}{4x - 1}$ is within a distance of 0.001 from 5?

12. Show that $\lim_{x \to 0} \dfrac{|x|}{x}$ does not exist.

13. Find functions f and g such that $\lim_{x \to 0} [f(x) + g(x)]$ exists but $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist. [Hint: See Example 8.]

PROBLEMS PLUS

1. Evaluate $\lim_{x \to 1} \dfrac{\sqrt{x} - 1}{\sqrt[3]{x} - 1}$ *Hint*: Introduce a new variable $t = \sqrt[6]{x}$.